Mathematical Analysis

...

Chapter 3: Differential Calculus & Series Expansion

...

Class Note Synopsis (Part 1)
B.Sc Semester 3
Subtopic: Function of a Single Variable

...

Real valued Function,
Injecction, Surjection, Bijection
& Inverse function (Introduction):
Appendix: Hyperbolic function

♦ *Debashis Chatterjee* ♦
♦ *Assistant Professor* ♦
♦ *Dept. of Statistics* ♦
♦ *Visva Bharati* ♦

♦ *This is Draft overview version of classnote* ♦ *November 16, 2021*

Instruction/ Suggestion:

This is Draft overview version of classnote. Such draft overview version of classnotes will be given time-to-time as a draft synopsis of the class discussions. Dear students, you should make your own "handwritten" classnote for your own future references & you are advised to write down in details in your own notebook and complete all home works (H.W) that will be given time-to-time (refer to class discussions for solutions and hints).

1 Real valued function

Definition 1.1. *Let A, B be two non-empty subsets of $\Re$. f is a rule through which each element of A is associated with* **unique** *element of B. Then f is called* **mapping** *from A to B.*

$$f : A \to B \quad \text{we write } y = f(x), x \in A, y \in B.$$

Here $A =$ domain set, $B =$ range set.

Range, codomain

range /image.
range is a subset of codomain.
**** Refer to class discussion.

Example 1.1.

Indicator fn. $f(x) = I_A(x), x \in \Re$, where

$$I_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

Dirichlet fn. $f(x) = I_Q(x), x \in \Re$, *where*

$$I_Q(x) = \begin{cases} 1 & if \quad x \in Q \\ 0 & if \quad x \in Q^c \end{cases}$$

Signum fn. $f(x) = sgn(x), x \in \Re$, *where*

$$I_A(x) = \begin{cases} \frac{|x|}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Identity fn. $f(x) = x \quad x \in \Re$

Modulus fn. $f(x) = |x| \quad x \in \Re$

Box fn. $f(x) = [x] \quad x \in \Re$ *(write down proper definition of box function by yourself! This was done in class 11)*

Step fn. $f(x) = \sum_{k=1}^{n} c_k I_{A_k}(x), \quad x \in \Re$, A_k *are piecewise disjoint intervals and* $\cup_k A_k = \Re$.

Even fn. $f(x), x \in \Re$ *is said to be even function if* $f(-x) = f(x)$.

Odd fn. $f(x), x \in \Re$ *is said to be odd function if* $f(-x) = -f(x)$.

Monotone increasing function. $f(x), x \in \Re$ *is said to be monotone increasing function if* $x_1 < x_2 \implies f(x_1) \leq f(x_2)$.

Monotone increasing function. $f(x), x \in \Re$ *is said to be monotone decreasing function if* $x_1 < x_2 \implies f(x_1) \geq f(x_2)$.

Hyperbolic function.

Hyperbolic functions occur in the calculations of angles and distances in hyperbolic geometry. In mathematics, hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle.
The hyperbolic cosine is the function:

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the hyperbolic sine is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Just as the points $(\cos t, \sin t)$ *form a circle with a unit radius, the points* $(\cosh t, \sinh t)$ *form the right half of the unit hyperbola.*
Also, just as the derivatives of $\sin(t)$ *and* $\cos(t)$ *are* $\cos(t)$ *and* $-\sin(t)$, *the derivatives of* $\sinh(t)$ *and* $\cosh(t)$ *are* $\cosh(t)$ *and* $+\sinh(t)$.

Animation: https://en.wikipedia.org/wiki/File:HyperbolicAnimation.gif, https://en.wikipedia.org/wiki/Hyperbolic_functions

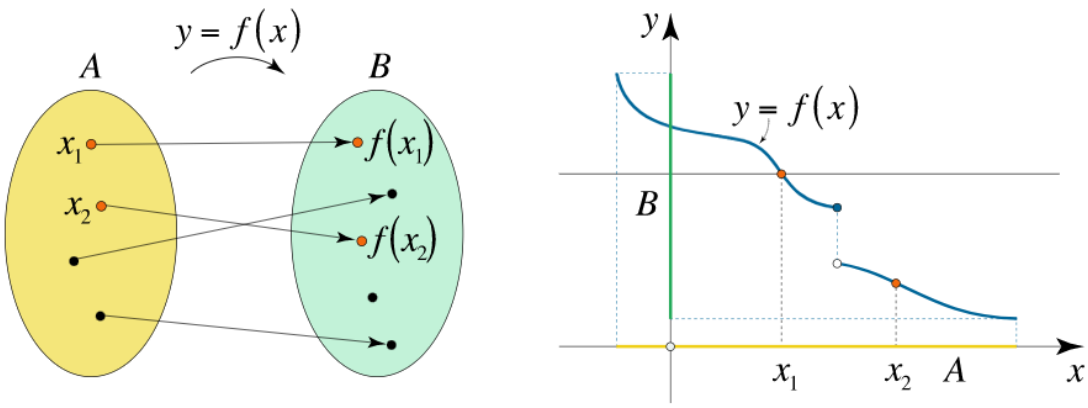


Figure 1: Injective function

2 Injection, Surjection, Bijection

Injections, surjections, and bijections are classes of functions distinguished by the manner in which arguments (from the domain) and images (output expressions from the codomain) are related or mapped to each other.

2.1 Injective Function

Definition 2.1. *The function is injective, or one-to-one, if each element of the codomain is mapped to by at most one element of the domain, or equivalently, if distinct elements of the domain map to distinct elements in the codomain. An injective function is also called an injection. Notationally:*

$$\forall x_1, x_2 \in A : \ x_1 \neq x_2 \ \Rightarrow f(x_1) \neq f(x_2) \, .$$

Horizontal Line test for injection

A horizontal line intersects the graph of an injective function at most once (that is, once or not at all). In this case, we say that the function passes the horizontal line test. If a horizontal line intersects the graph of a function in more than one point, the function fails the horizontal line test and is not injective.
A horizontal line intersects the graph of an injective function at most once (that is, once or not at all). In this case, we say that the function passes the horizontal line test.

If a horizontal line intersects the graph of a function in more than one point, the function fails the horizontal line test and is not injective.

2.2 Surjection/ Surjective function/ Onto function

Definition 2.2. *A function f from A to B is called surjective (or onto) if for every y in the codomain B there exists at least one x in the domain A such that:*

$$\forall y \in B : \ \exists x \in A \ such\ that\ y = f(x) \, .$$

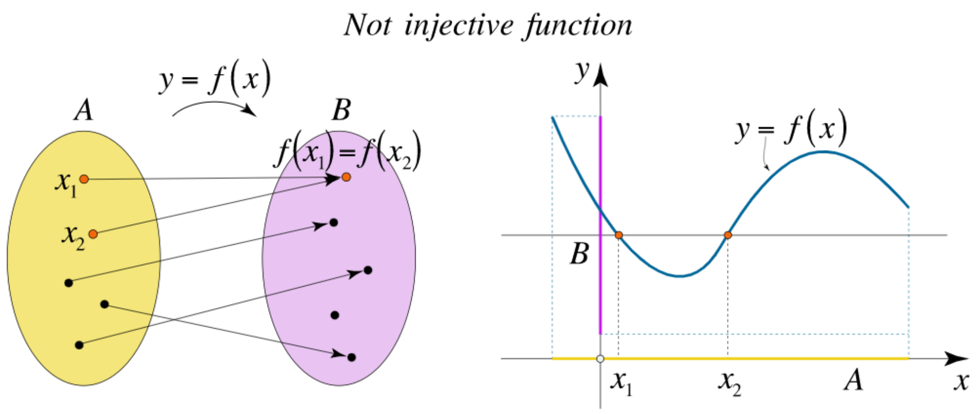


Figure 2: Not injective function

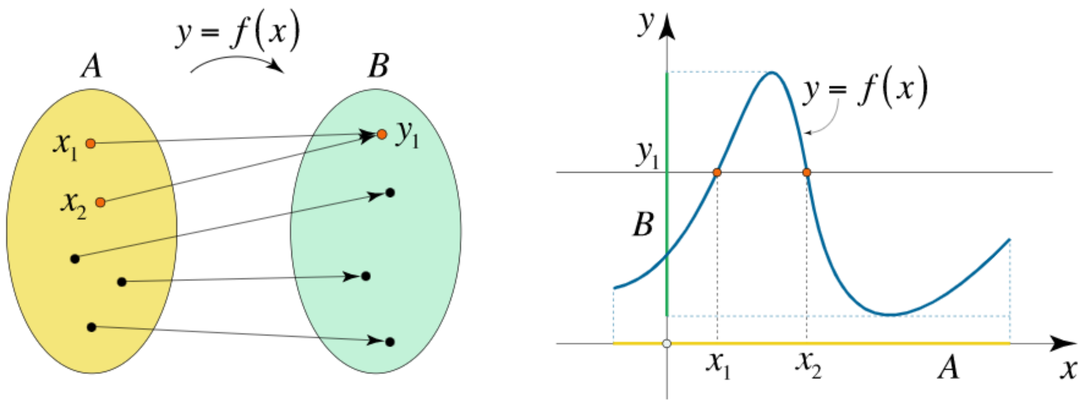


Figure 3: Surjective function

Horizontal Line test for surjection

The range and the codomain for a surjective function are identical.
Any horizontal line should intersect the graph of a surjective function at least once (once or more).

2.3 Bijective function /One-to-one correspondence

Definition 2.3. *A function f from A to B is called bijection (one-one + onto / one to one correspondence) if for every y in the codomain B there exists* **EXACTLY** *one x in the domain A such that:*

$$\forall y \in B : \ \exists! x \in A \text{ such that } y = f(x).$$

Horizontal Line test for Bijection

The range and the codomain for a surjective function are identical.

Any horizontal line passing through any element of the range should intersect the graph of a bijective function exactly once.

A Appendix: More about Hyperbolic Functions

Definition A.1. *The hyperbolic cosine is the function:*

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the hyperbolic sine is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

. The other hyperbolic functions are:

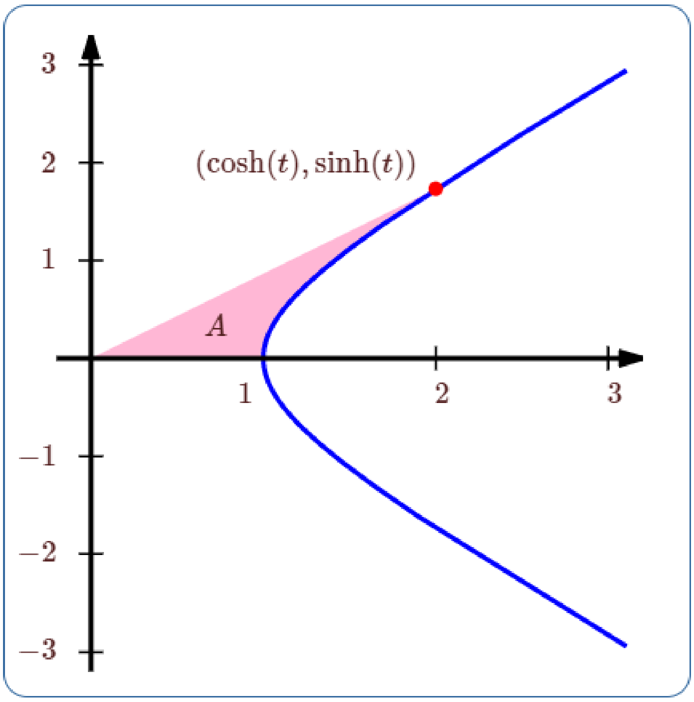
$$\begin{aligned} \tanh x &= \tfrac{\sinh x}{\cosh x} \\ \coth x &= \tfrac{\cosh x}{\sinh x} \\ sechx &= \tfrac{1}{\cosh x} \\ cschx &= \tfrac{1}{\sinh x} \end{aligned}$$

Theorem 1. *The range of* $\cosh x$ *is* $[1, \infty)$

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned} y &= \tfrac{e^x+e^{-x}}{2} \\ 2y &= e^x + e^{-x} \\ 2ye^x &= e^{2x} + 1 \\ 0 &= e^{2x} - 2ye^x + 1 \\ e^x &= \tfrac{2y \pm \sqrt{4y^2-4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \end{aligned}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.
Now suppose $y \geq 1$. So, $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number and $y = \cosh x$. Hence, y is in the range of $\cosh(x)$. □



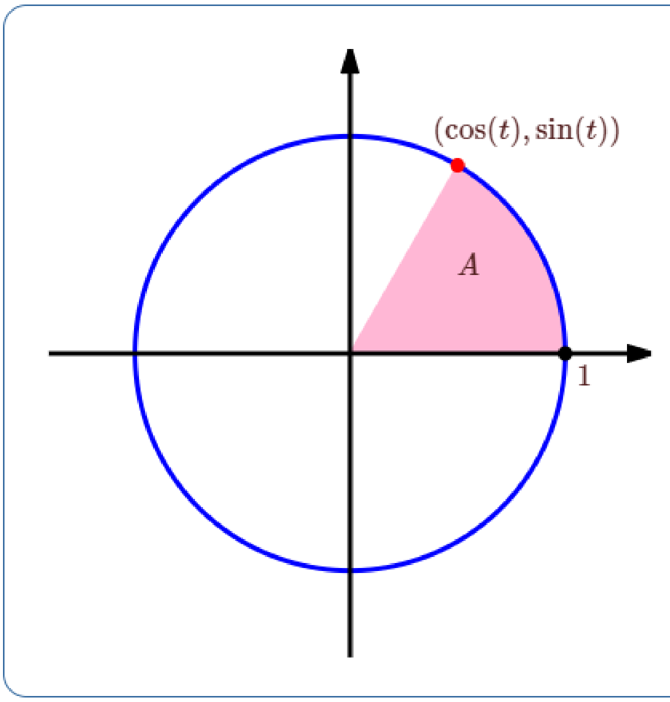


Figure 4: Geometric definitions of sin, cos, sinh, cosh: Here t is twice the shaded area in each figure. (Prove that, (H.W)

H. W.

(1) Show that, The domain of $\coth x$ and $cschx$ is $x \neq 0$ while the domain of the other hyperbolic functions is all real numbers.
(2)show that if $y = \sinh x$, then $x = \ln(y + \sqrt{y^2 + 1})$. Using this (or may not), Show that the range of $\sinh x$ is all real numbers.
(3) Show that the range of $tanhx$ is $(-1, 1)$. What are the ranges of $cothx, sechx, andcschx$? (Hint: Use the fact that they are reciprocal functions.)

Theorem 2. *For all* $x \in \Re$, $\cosh^2 x - \sinh^2 x = 1$

Proof.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

□

H.W

Show the following in a similar manner:

$$1 - tanh^2 x = sech^2 x \qquad \text{and} \qquad coth^2 x - 1 = csch^2 x.$$

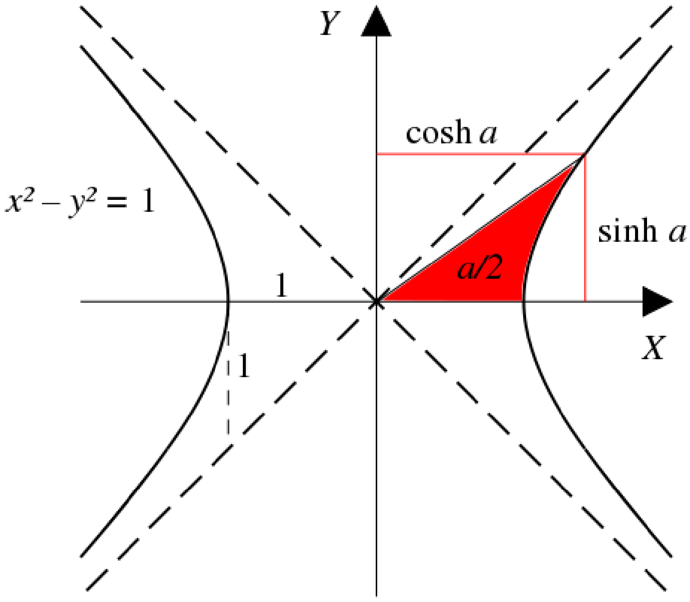


Figure 5: A ray through the unit hyperbola $x^2 - y^2 = 1$ at the point $(\cosh a, \sinh a)$, where a is twice the area between the ray, the hyperbola, and the x-axis. For points on the hyperbola below the x-axis, the area is considered negative. Reference: https://www.whitman.edu/mathematics/calculus_online/section04.11.html

B References

1. Apostol: Mathematical Analysis
2. Wekepedia reference. https://en.wikipedia.org/wiki/Bijection,_injection_and_surjection
3. https://math24.net/injection-surjection-bijection.html